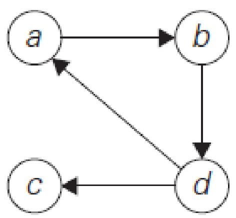
WARSHALL ALGORITHM

- The adjacency matrix $A = \{a_{ij}\}$ of a directed graph is the boolean matrix that has 1 in its ith row and jth column if and only if there is a directed edge from the ith vertex to the jth vertex.
- The matrix containing the information about the existence of directed paths between any two vertices of a given graph is called the transitive closure of the digraph,
- The ***transitive closure*** of a directed graph with n vertices can be defined as the $n \times n$ boolean matrix $T = \{t_{ij}\}$, in which the element in the ith row and the jth column is 1 if there exists a path of a positive length from the ith vertex to the jth vertex; otherwise, t_{ij} is 0.



(a)

$$A = \begin{array}{c|cccc|} & a & b & c & d \\ a & 0 & 1 & 0 & 0 \\ b & 0 & 0 & 0 & 1 \\ c & 0 & 0 & 0 & 0 \\ d & 1 & 0 & 1 & 0 \end{array}$$

(b)

$$T = \begin{array}{c|cccc|} & a & b & c & d \\ a & 1 & 1 & 1 & 1 \\ b & 1 & 1 & 1 & 1 \\ c & 0 & 0 & 0 & 0 \\ d & 1 & 1 & 1 & 1 \end{array}$$

(c)

(a) Digraph. (b) Its adjacency matrix. (c) Its transitive closure.

- Warshall Algorithm can be used to find the **transitive closure** for the given digraph.
- Warshall's algorithm constructs the transitive closure through a series of $n \times n$ boolean matrices:

$$R^{(0)}, \ldots, R^{(k-1)}, R^{(k)}, \ldots R^{(n)}.$$

- The element $r^{(k)}_{ij}$ in the ith row and jth column of matrix $R^{(k)}$ $(i, j) = 1, 2, \ldots, n, k = 0, 1, \ldots, n)$ is equal to 1 if and only if there exists a directed path of a positive length from the ith vertex to the jth vertex with each intermediate vertex, if any, numbered not higher than k.
- $R^{(0)}$ is nothing other than the adjacency matrix of the digraph.
- $R^{(1)}$ contains the information about paths that can use the first vertex as intermediate vertex.
- $R^{(n)}$, reflects paths that can use all n vertices of the digraph as intermediate and hence is nothing other than the digraph's transitive closure.

- The formula for generating the elements of matrix $R^{(k)}$ from the elements of matrix $R^{(k-1)}$:

$$r_{ij}^{(k)} = r_{ij}^{(k-1)} \quad \text{or} \quad \left(r_{ik}^{(k-1)} \text{ and } r_{kj}^{(k-1)}\right)$$

- This formula implies the following rule for generating elements of matrix $R^{(k)}$ from elements of matrix $R^{(k-1)}$
  - ❖ If an element r_{ij} is 1 in $R^{(k-1)}$, it remains 1 in $R^{(k)}$.
  - ❖ If an element r_{ij} is 0 in $R^{(k-1)}$, it has to be changed to 1 in $R^{(k)}$ if and only if the element in its row i and column k and the element in its column j and row k are both 1's in $R^{(k-1)}$
- Here is pseudocode of Warshall's algorithm.
- **ALGORITHM** *Warshall(A*[1..*n*, 1..*n*]*)*

//ImplementsWarshall' s algorithm for computing the transitive closure

//Input: The adjacency matrix *A* of a digraph with *n* vertices

//Output: The transitive closure of the digraph

$R^{(0)} \leftarrow A$

for $k \leftarrow 1$ **to** n **do**

  for $i \leftarrow 1$ **to** n **do**

    for $j \leftarrow 1$ **to** n **do**

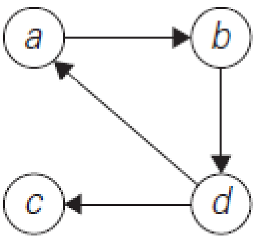
      $R^{(k)}[i, j] \leftarrow R^{(k-1)}[i, j]$ **or** $(R^{(k-1)}[i, k]$ **and** $R^{(k-1)}[k, j])$

return $R^{(n)}$

- The time efficiency is only $\Theta(n^3)$.

PROBLEM

Find the transitive closure for the given graph



The adjacency matrix for the given graph is

$$R^{(0)} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{cccc} a & b & c & d \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}$$

To find $R^{(1)}$, keeping the first vertex 'a' as intermediate vertex

$R^{(1)}[b,b] = R^{(0)}[b,b]$ or $\{ R^{(0)}[b,a]$ and $R^{(0)}[a,b]\} = 0$ or $\{0$ and $1\}= 0$

$R^{(1)}[b,c] = R^{(0)}[b,c]$ or $\{ R^{(0)}[b,a]$ and $R^{(0)}[a,c]\} = 0$ or $\{0$ and $0\}= 0$

$R^{(1)}[c,b] = R^{(0)}[c,b]$ or $\{ R^{(0)}[c,a]$ and $R^{(0)}[a,b]\} = 0$ or $\{0$ and $1\}= 0$

$R^{(1)}[c,c] = R^{(0)}[c,c]$ or $\{ R^{(0)}[c,a]$ and $R^{(0)}[a,c]\} = 0$ or $\{0$ and $0\}= 0$

$R^{(1)}[c,d] = R^{(0)}[c,d]$ or $\{ R^{(0)}[c,a]$ and $R^{(0)}[a,d]\} = 0$ or $\{0$ and $0\}= 0$

$R^{(1)}[d,b] = R^{(0)}[d,b]$ or $\{ R^{(0)}[d,a]$ and $R^{(0)}[a,b]\} = 0$ or $\{1$ and $1\}= 1$

$R^{(1)}[d,d] = R^{(0)}[d,d]$ or $\{ R^{(0)}[d,a]$ and $R^{(0)}[a,d]\} = 0$ or $\{1$ and $0\}= 0$

$$R^{(1)} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{cccc} a & b & c & d \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & \mathbf{1} & 1 & 0 \end{array}$$

To find $R^{(2)}$, keeping the second vertex 'b' as intermediate vertex

$R^{(2)}[a,a] = R^{(1)}[a,a]$ or $\{ R^{(1)}[a,b]$ and $R^{(1)}[b,a]\} = 0$ or $\{1$ and $0\}= 0$

$R^{(2)}[a,c] = R^{(1)}[a,c]$ or $\{ R^{(1)}[a,b]$ and $R^{(1)}[b,c]\} = 0$ or $\{1$ and $0\}= 0$

$R^{(2)}[a,d] = R^{(1)}[a,d]$ or $\{ R^{(1)}[a,b]$ and $R^{(1)}[b,d]\} = 0$ or $\{1$ and $1\}= 1$

$R^{(2)}[c,a] = R^{(1)}[c,a]$ or $\{ R^{(1)}[c,b]$ and $R^{(1)}[b,a]\} = 0$ or $\{0$ and $0\}= 0$

$R^{(2)}[c,c] = R^{(1)}[c,c]$ or $\{ R^{(1)}[c,b]$ and $R^{(1)}[b,c]\} = 0$ or $\{0$ and $0\}= 0$

$R^{(2)}[c,d] = R^{(1)}[c,d]$ or $\{ R^{(1)}[c,b]$ and $R^{(1)}[b,d]\} = 0$ or $\{0$ and $1\}= 0$

$R^{(2)}[d,d] = R^{(1)}[d,d]$ or $\{ R^{(1)}[d,b]$ and $R^{(1)}[b,d]\} = 0$ or $\{1$ and $1\}= 1$

$$R^{(2)} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{cccc} a & b & c & d \\ \hline 0 & 1 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & \mathbf{1} \end{array}$$

To find $R^{(3)}$, keeping the third vertex 'c' as intermediate vertex

$R^{(3)}[a,a] = R^{(2)}[a,a]$ or { $R^{(2)}[a,c]$ and $R^{(2)}[c,a]$} = 0 or {0 and 0}= 0

$R^{(3)}[b,a] = R^{(2)}[b,a]$ or { $R^{(2)}[b,c]$ and $R^{(2)}[c,a]$} = 0 or {0 and 0}= 0

$R^{(3)}[b,b] = R^{(2)}[b,b]$ or { $R^{(2)}[b,c]$ and $R^{(2)}[c,b]$} = 0 or {0 and 0}= 0

$$R^{(3)} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

To find $R^{(4)}$, keeping the fourth vertex 'd' as intermediate vertex

$R^{(4)}[a,a] = R^{(3)}[a,a]$ or { $R^{(3)}[a,d]$ and $R^{(3)}[d,a]$} = 0 or {1 and 1}= 1

$R^{(4)}[a,c] = R^{(3)}[a,c]$ or { $R^{(3)}[a,d]$ and $R^{(3)}[d,c]$} = 0 or {1 and 1}= 1

$R^{(4)}[b,a] = R^{(3)}[b,a]$ or { $R^{(3)}[b,d]$ and $R^{(3)}[d,a]$} = 0 or {1 and 1}= 1

$R^{(4)}[b,b] = R^{(3)}[b,b]$ or { $R^{(3)}[b,d]$ and $R^{(3)}[d,b]$} = 0 or {1 and 1}= 1

$R^{(4)}[b,c] = R^{(3)}[b,c]$ or { $R^{(3)}[b,d]$ and $R^{(3)}[d,c]$} = 0 or {1 and 1}= 1

$R^{(4)}[c,a] = R^{(3)}[c,a]$ or { $R^{(3)}[c,d]$ and $R^{(3)}[d,a]$} = 0 or {0 and 1}= 0

$R^{(4)}[c,b] = R^{(3)}[c,b]$ or { $R^{(3)}[c,d]$ and $R^{(3)}[d,b]$} = 0 or {0 and 1}= 0

$R^{(4)}[c,c] = R^{(3)}[c,c]$ or { $R^{(3)}[c,d]$ and $R^{(3)}[d,c]$} = 0 or {0 and 1}= 0

$$R^{(4)} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} \mathbf{1} & 1 & \mathbf{1} & 1 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

The transitive closure for the given graph is

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} \mathbf{1} & 1 & \mathbf{1} & 1 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$